

NPTEL ONLINE CERTIFICATION COURSES

Management of Fixed Income Securities

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Lecture 41: Estimation of Binomial Trees-I

CONCEPTS COVERED

- Sub-division of binomial trees
- Approaches for Estimation of Binomial Interest Rate Movements

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KEYWORDS

- **Binomial Process**
- **Binomial distribution**

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Subdividing the Binomial Tree

- The binomial model is more realistic when we subdivide the periods to maturity into a number of sub-periods.

- As the number of sub-periods increase:
  - The length of each period becomes smaller, making the assumption that the spot rate will either increase or decrease more plausible.

  - The number of possible rates at maturity increases, which again adds realism to the model.

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Subdividing the Binomial Tree Example

- Instead of using a three-period binomial tree, where the length of each period is a year, suppose we evaluate the bond using a six-period tree with the length of each period being six months.
- If we do this, we need to:
  - Divide the one-year spot rates and the annual coupon by two.
  - Adjust the u and d parameters to reflect changes over a six-month period instead of one year.
  - Define the binomial tree of spot rates for five periods, each with a length of six months.
- In general, let:

  h = length of the period in years

  n = number of periods of length h defining the maturity of the bond

  where n = (maturity in years)/h

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Subdividing the Binomial Tree Example

- For a three-year bond evaluated over ***quarterly*** periods:
  h = 1/4 of a year
  Maturity = n = (3 years)/(1/4 years) = 12 periods
  Binomial interest rates tree with n-1 = 12-1 = 11 periods

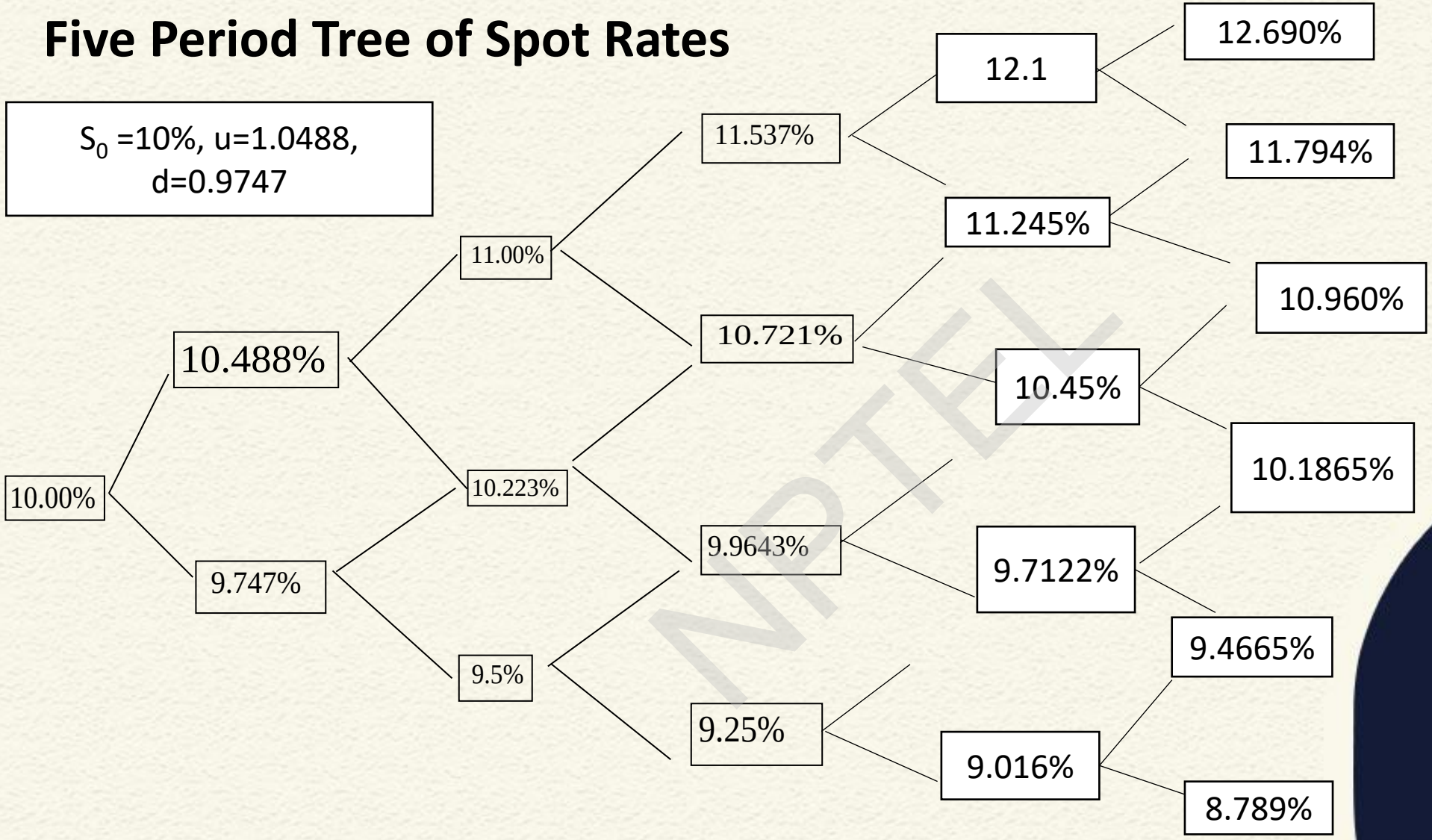
- For a three-year bond evaluated over ***monthly*** periods:
  h = 1/12 of a year
  Maturity = n = (3 years)/(1/12 years) = 36 periods
  35-period binomial tree of spot rates

- For a three-year bond evaluated over ***weekly*** periods
  h = 1/52 of a year
  Maturity = 156 periods of length one week
  155-period binomial tree of spot rates

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Five Period Tree of Spot Rates

S_0 =10%, u=1.0488, d=0.9747



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Approaches for Estimation of Binomial Interest Rate Movements

- There are two general approaches to estimate binomial interest rate movements.
  - Estimating the u and d Parameters – Equilibrium Model developed by Rendelman and Bartter (1980) and Cox, Ingersoll, and Ross (1985)
  - Calibration Model – Arbitrage-Free Model developed by Black, Derman, and Toy (1990), Ho and Lee (1986), and Heath, Jarrow, and Morton (1992)

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u and d Estimation Approach – Equilibrium Model

- This approach solves for the u and d values that equate the mean and variance of the distribution of the logarithmic return to their empirical values.
- Bond values obtained using this technique are then compared to their market prices to determine if mispricing occurs.
- Because the model's values and market prices are often different, additional assumptions regarding risk premiums must be made to explain market prices.

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Estimating u and d

- The estimating equations for determining u and d are obtained by mathematically solving for the u and d values that make the statistical characteristics of a binomial distribution of the stock's logarithmic returns equal to the characteristic's estimated value.

- As background to understand this approach, let us first examine the probability distribution that characterizes a binomial process.

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Binomial Process

- The binomial process that we have described for spot rates yields after n periods a distribution of n + 1possible spot rates.
- This distribution is not normally distributed because the left-side of the distribution has a limit at zero (i.e. we generally do not have negative spot rates).
- The distribution of spot rate can be converted into a distribution of logarithmic returns, g_n:

$$g_n = \ln\left[\frac{S_n}{S_0}\right]$$

- The distribution of logarithmic returns can take on negative values and can be normally distributed if the probability of the spot rate increasing in one period (q) is 0.5. (q is the probability value)

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Binomial Process Explanations

- When n = 1, there are two possible spot rates and logarithmic returns:

$$g_u = \ln \frac{uS_0}{S_0} = \ln(u) = \ln(1.1) = 0.095$$

$$g_d = \ln \frac{dS_0}{S_0} = \ln(d) = \ln(0.95) = -0.0513$$

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Binomial Process Explanations

When n = 2, there are three possible spot rates and logarithmic returns:

$$g_{uu} = \ln \frac{u^2 S_0}{S_0} = \ln(u^2) = \ln(1.1^2) = 0.1906$$

$$g_{ud} = \ln \frac{udS_0}{S_0} = \ln(ud) = \ln[(1.1)(0.95)] = 0.044$$

$$g_{dd} = \ln \frac{d^2 S_0}{S_0} = \ln(d^2) = \ln(0.95^2) = -0.1026$$

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Binomial Process Explanations

When n = 3, there are four possible spot rates and logarithmic returns:

$$g_{uuu} = \ln \frac{u^3 S_0}{S_0} = \ln(u^3) = \ln(1.1^3) = 0.2859$$

$$g_{uud} = \ln \frac{u^2 d S_0}{S_0} = \ln(u^2 d) = \ln[(1.1^2)(0.95)] = 0.1393$$

$$g_{udd} = \ln \frac{u\ d^2 S_0}{S_0} = \ln(u\ d^2) = \ln[(1.1)(0.95^2)] = -0.0073$$

$$g_{ddd} = \ln \frac{d^3 S_0}{S_0} = \ln(d^3) = \ln(0.95^3) = -0.1539$$

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Binomial Process

- n = 1, there are two possible rates and logarithmic returns
- n = 2, there are three possible rate and logarithmic returns
- n = 3, there are four possible rates and logarithmic returns

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Binomial Process

- The probability of attaining any of these rates is equal to the probability of the spot rate increasing j times in n period: p_{nj}.
- For example: the probability of attaining spot rate 10.45% in period 2 is equal to the probability of the spot rate increasing one time (j=1) in two periods (n=2), p_{21}.
- In a binomial process, this probability is

$$p_{nj} = \frac{n!}{(n-j)!\,j!} q^{j} (1-q)^{n-j}$$

Where, q is the probability

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Binomial Distribution

- In this example after two periods, the probability of spot rate equal to 9.53% is p_{11}= 0.50 and -5.13% is p_{10}= 0.50. Using the binomial probabilities, the expected value and variance of the logarithmic return after one period are :

$$E(g_1) = 0.5(0.0953) + 0.5(-0.0513) = 0.022$$

$$V(g_1) = 0.25[0.0953 - 0.022]^2 + 0.5[-0.0513 - 0.022]^2 = 0.0054$$

- The expected value and variance of the logarithmic return after two periods are 0.044 and 0.0108:

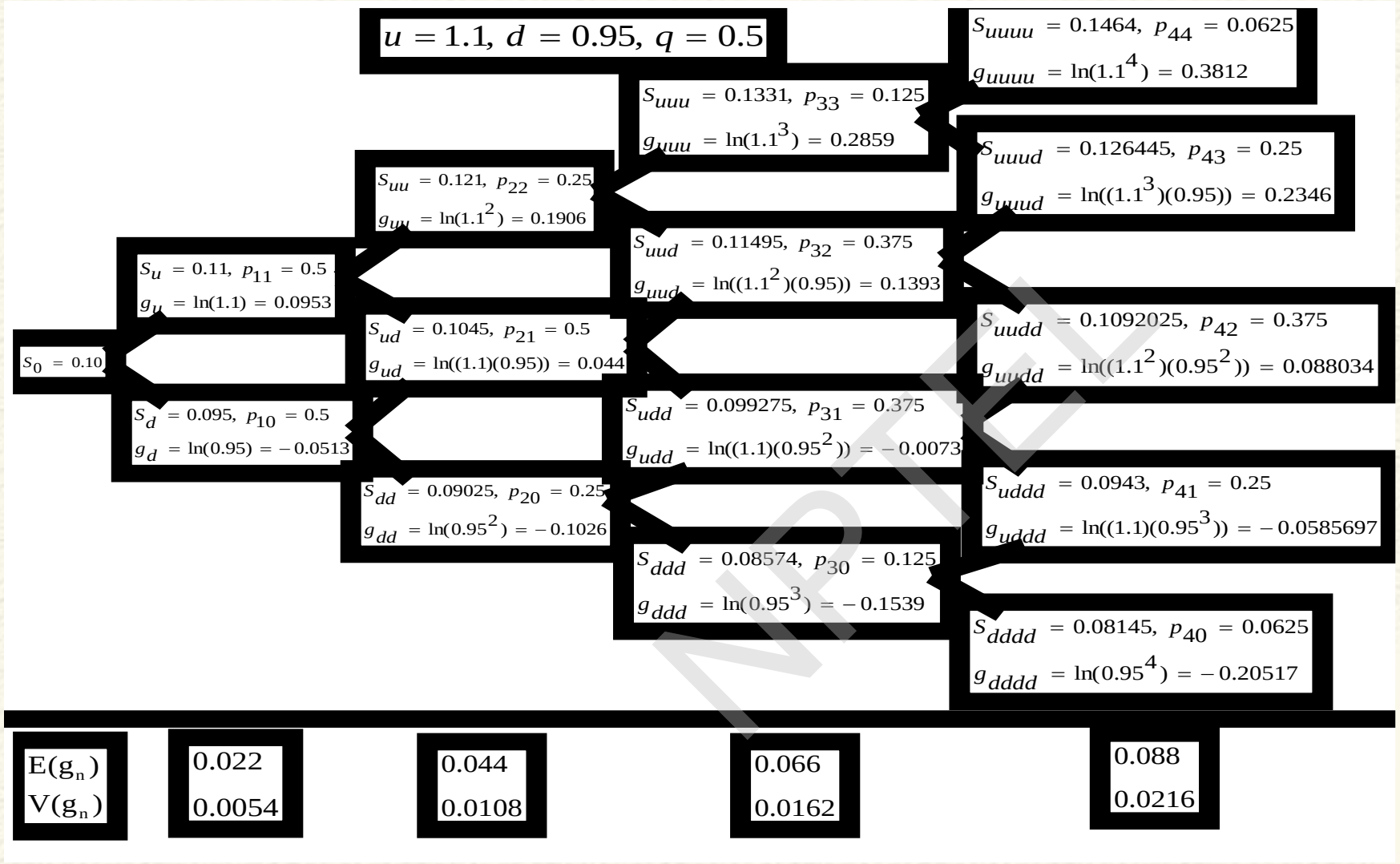
$$E(g_2) = 0.25(0.1906) + 0.5(0.0440) + 0.25(-0.1026) = 0.044$$

$$V(g_2) = 0.25[0.1906 - 0.044]^2 + 0.5[0.044 - 0.044]^2 + 0.25[-0.1026 - 0.044]^2 = 0.0108$$

- The expected value and variance of the logarithmic return after three periods are 0.066 and 0.0162
- The expected value and variance of the logarithmic return after four periods are 0.088 and 0.0216

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Binomial Process and Distribution Example



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Binomial Distribution

- In examining each distribution's mean and variance, note that as the number of periods increases, the expected value and variance increase by a multiplicative factor such that:

$$E(g_n) = nE(g_1)$$

$$V(g_n) = nV(g_1)$$

- The parameter values (expected value and variance) after n periods are equal to the parameter values for one period time the number of periods.
- The expected value and the variance are also equal to

$$E(g_n) = n[q \ln u + (1-q) \ln d]$$

$$V(g_n) = nq(1-q)[\ln(u/d)]^2$$

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CONCLUSIONS

- The binomial model becomes more realistic when we subdivide the periods to maturity into a number of sub-periods
- There are two general approaches to estimate binomial interest rate movements: (i) Equilibrium model and (ii) Calibration model
- The equilibrium model solves for the u and d values that equate the mean and variance of the distribution of the logarithmic return to their empirical values

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REFERENCES

- Johnson, S. R (2010): Bond Evaluation, Selection and Management, John Wiley & Sons, 2nd Edition.
- Fabozzi, J. Frank and Mann, V. Steven (2005): The Hand Book of Fixed Income Securities, Tata McGraw-Hill, 7th Edition.

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Lecture 42: Estimation of Binomial Trees-II

CONCEPTS COVERED

- Solving the Values of u and d
- Estimation of Annualized Mean and Variance of log returns

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KEYWORDS

- Annualized Mean
- Annualized Variance

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Solving the Values of u and d

- Given the features of a binomial distribution, the formulas for estimating u and d are found by solving for the u and d values that make the expected value and the variance of the binomial distribution of the logarithmic return of spot rates equal to their respective estimated parameter values under the assumption that q = 0.5 (or the distribution is normal).
- If we let μ_e and V_e be the estimated mean and variance of the logarithmic return of spot rates for a period equal in length to n periods, then our objective is to solve for the u and d values that simultaneously satisfy the following equations:

$$nE(g_1) = n[q \ln u + (1-q) \ln d] = \mu_e$$

$$nV(g_1) = nq(1-q)[\ln(u/d)]^2 = V_e$$

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Solving the Values of u and d

- If q = 0.5, then the formula values for u and d that satisfy the two equations are

$$u = e^{\sqrt{V_e / n} + \mu_e / n}$$

$$d = e^{-\sqrt{V_e / n} + \mu_e / n}$$

- If the *estimated* expected value and variance of the logarithmic return were $\mu_e = 0.044$ and $V_e = 0.0108$ for a period equal in length to n = 2, then using the above equations, u would be 1.1 and d would be 0.95:

$$u = e^{\sqrt{0.0108/2} + 0.044/2} = 1.1$$

$$d = e^{-\sqrt{0.0108/2} + 0.044/2} = 0.95$$

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u and d for Large n

- In the equations for u and d, as n increases the mean term in the exponent goes to zero quicker than the square root term.

- For large n (e.g., n = 30), the mean term's impact on u and d is negligible and u and d can be estimated as:

$$u = e^{\sqrt{V_e / n}}$$

$$d = e^{-\sqrt{V_e / n}} = 1/u$$

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Annualized Mean and Variance

- μ_e and V_e are the mean and variance for a length of time equal to the bond's maturity.
- Often the annualized mean and variance are used.
- The annualized mean and variance are obtained by multiplying the estimated mean and variance of a given length (e.g., month) by the number of periods of that length in a year (e.g., 12).
- For example: if half yearly data is used to estimate the mean and variance, then we simply multiply those estimates by two to obtain the annualized parameters

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Annualized Mean and Variance

- When the annualized mean and variance are used, then these parameters must by multiplied by the proportion h, as the time of the period being analyzed expressed as a proportion of a year, and n is not needed since h defines the length of tree's period:

$$u = e^{\sqrt{hV_e^A} + h\mu_e^A}$$

$$d = e^{-\sqrt{hV_e^A} + h\mu_e^A}$$

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Annualized Mean and Variance

- If the annualized mean and variance of the logarithmic return of one-year spot rates were 0.044 and 0.0108, and we wanted to evaluate a three-year bond with six-month periods (h = ½ of a year), then we would use a six-period tree to value the bond (n = (3 years)/(½) = 6 periods) and u and d would be 1.1 and 0.95:

$$u = e^{\sqrt{(1/2)0.0108} + (1/2)0.044} = 1.1$$

$$d = e^{-\sqrt{(1/2)0.0108} + (1/2)0.044} = 0.95$$

- If we make the length of the period monthly (h=1/12), then we would value the three-year bond with a 36-period tree and u and d would be :

$$u = e^{\sqrt{(1/12)0.0108} + (1/12)0.044} = 1.03424$$

$$d = e^{-\sqrt{(1/12)0.0108} + (1/12)0.044} = 0.9740$$

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Annualized Mean and Variance

- The ***annualized standard deviation*** cannot be obtained simply by multiplying the quarterly standard deviation by four.

- One must first multiply the quarterly variance by four and then take the square root of the resulting annualized variance.

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Estimating μ_e and V_e

- The simplest estimate of μ_e and V_e is the average mean and variance computed from an historical sample of spot rates.
- ***Example:***
  - Historical quarterly one-year spot rates over 13 quarters are available
  - The 12 logarithmic returns are calculated by taking the natural log of the ratio of spot rates in one period to the rate in the previous period (S_t/S_{t-1}).

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Estimating μ_e and V_e

Quarter	Spot Rate, S_t	S_t/S_{t-1}	$g_t = \ln(S_t/S_{t-1})$	$(g_t - \mu_e)^2$
1	10.60%	–	--	–
2	10.00%	0.9434	-0.06	0.0034
3	9.40%	0.94	-0.06	0.00383
4	8.80%	0.9362	-0.07	0.00435
5	9.40%	1.0682	0.066	0.00435
6	10.00%	1.0638	0.062	0.00383
7	10.60%	1.06	0.058	0.0034
8	10.00%	0.9434	-0.06	0.0034
9	9.40%	0.94	-0.06	0.00383
10	8.80%	0.9362	-0.07	0.00435
11	9.40%	1.0682	0.066	0.00435
12	10.00%	1.0638	0.062	0.00383
13	10.60%	1.06	0.058	0.0034
			0	0.0463
			$\mu_e = 0$	

$$V_e^q = \frac{.046297}{11} = .004209$$

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Estimating μ_e and V_e

- From this data, the historical quarterly logarithmic mean return and variance are:

$$\mu_e = \frac{\sum_{t=1}^{12} g_t}{12} = \frac{0}{12} = 0$$

$$V_e = \frac{\sum_{t=1}^{12} [g_t - \mu_e]^2}{11} = \frac{0.046297}{11} = 0.004209$$

- Multiplying the historical quarterly mean and variance by 4, we obtain an *annualized mean* and *variance*, respectively, of 0 and 0.016836.

$$\mu_e^A = 4\mu_e^q$$
$$\mu_e^A = 4(0)$$
$$\mu_e^A = 0$$

$$V_e^A = 4V_e^q$$
$$V_e^A = 4(0.004209)$$
$$V_e^A = 0.016836$$

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Estimating μ_e and V_e

- Given the estimated annualized mean and variance, u and d can be estimated once we determine the number of periods to subdivide.

Length	h	u	d
Year	1	1.1385	0.8783
Quarter	1/4	1.0670	0.9372
Month	1/12	1.0382	0.9632

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Estimating μ_e and V_e

- A binomial interest rates tree generated using the u and d estimation approach is constrained to have an end-of-the-period distribution with a mean and variance that matches the analyst's estimated mean and variance.
- The binomial tree generated from u and d estimates is not constrained to yield a bond price that matches its equilibrium price: price obtain by discounting the bond's cash flows by spot rates.
- Analysts using such models need to make additional assumptions about the risk premium in order to explain the bond's equilibrium price.
- In contrast, calibration models are constrained to match the current term structure of spot rates and therefore yield bond prices that are equal to their equilibrium values.

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CONCLUSIONS

- Given the features of a binomial distribution, the formulas for estimating u and d are found by solving for the u and d values that make the expected value and the variance of the binomial distribution of the logarithmic return of spot rates equal to their respective estimated parameter values under the assumption that the distribution is normal.
- The annualized mean and variance are obtained by multiplying the estimated mean and variance of a given length by the number of periods of that length in a year.

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REFERENCES

- Johnson, S. R (2010): Bond Evaluation, Selection and Management, John Wiley & Sons, 2nd Edition.
- Fabozzi, J. Frank and Mann, V. Steven (2005): The Hand Book of Fixed Income Securities, Tata McGraw-Hill, 7th Edition.

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Lecture 43: Estimation of Binomial Trees-III

CONCEPTS COVERED

- **Calibration Model**

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KEYWORDS

- **Variability condition**
- **Price condition**

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Calibration Model

- The calibration model generates a binomial tree by finding spot rates that satisfy two conditions:
  - A ***variability condition*** that governs the relation between the upper and lower spot rates.
  - A ***price condition*** in which the bond value obtained from the tree is consistent with the equilibrium bond price given the current spot yield curve.

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Variability Condition

- In our derivation of the formulas for u and d, we assumed that the distribution of the logarithmic return of spot rates was normal.

- This assumption also implies the following relationship between the upper and lower spot rate:

$$S_u = S_d e^{2\sqrt{V_e / n}}$$

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Variability Condition

Given: $S_u = uS_0$

$S_d = dS_0$

Therefore: $\frac{S_u}{u} = S_0 = \frac{S_d}{d}$

$S_u = S_d \frac{u}{d}$

Substituting the equations for u and d, we obtain:

$$S_u = S_d \frac{e^{\sqrt{V_e/n} + \mu_e/n}}{e^{-\sqrt{V_e/n} + \mu_e/n}} = S_d e^{2\sqrt{V_e/n}}$$

In terms of the annualized variance: $S_u = S_d e^{2\sqrt{hV_e^A}}$

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Variability Condition Example

- Given a lower rate of 9.5% and an annualized variance of 0.0054, the upper rate for a one-period binomial tree of length one year (h = 1) would be :

$$S_u = 9.5\% e^{2\sqrt{.0054}} = 0.11 = 11\%$$

- If the current one-year spot rate were 10%, then these upper and lower rates would be consistent with the upward and downward parameters of u = 1.1 and d = 0.95.

- This variability condition would therefore result in a binomial tree identical to the one shown in the earlier example.

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Price Condition

- A ***price condition*** requires that the bond value obtained from the tree is equal to the equilibrium bond price given the current spot yield curve.

- To generate a tree that satisfies the price condition requires solving for a lower spot rate that along with the variability relation yields a bond price that is equal to the equilibrium bond price.

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Price Condition

- Suppose the current spot yield curve has the following one-, two-, and three-year spot rates (y_t):

$$y_1 = 10\%$$
$$y_2 = 10.12238\%$$
$$y_3 = 10.24488\%$$

- Assume the estimated annualized logarithmic mean and variance are:

$$\mu_e^A = 0.048167$$
$$V_e^A = 0.0054$$

- ***Using the u and d approach***, a one-period tree of length one year would have up and down parameters of (i.e u and d) would be:

$$u = e^{\sqrt{(1)0.0054} + (1)0.048167} = 1.12936$$

$$d = e^{-\sqrt{(1)0.0054} + (1)0.048167} = 0.975$$

- Given the current one-period spot rate of 10%, the tree's possible spot rate would be S_u = 11.2936% and S_d = 9.75%.
- These rates are not consistent with the existing term structure.

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Price Condition

- If we value a two-year zero coupon bond with a face value of Rs. 1 using this tree, we obtain a value of 0.82258 that, given the two-year spot rate of 10.12238%, differs from the equilibrium price on the two-year zero discount bond of B_0^M = 0.8246:

$$B_0 = \frac{0.5B_u + 0.5B_d}{1+S_0} = \frac{0.5[1/1.112936] + 0.5[1/1.0975]}{1.10} = 0.82258$$

$$B_0^M = \frac{1}{(1+y_2)^2} = \frac{1}{(1.1012238)^2} = 0.8246$$

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Calibration of Binomial Tree to Two Period Zero Coupon Bond with Face Value of Rs.1

$$S_u = 0.0975\, e^{2\sqrt{0.0054}} = 0.112936$$

$$B_u = \frac{1}{1.112936} = 0.898524$$

$$B_{uu} = 1$$

$$S_0 = 0.10$$

$$B_0 = \frac{0.5\,(0.898524) + 0.5(0.9111617)}{1.10}$$

$$B_0 = 0.82258$$

$$B_{ud} = 1$$

$$S_d = 0.0975$$

$$B_d = \frac{1}{1.0975} = 0.9111617$$

$$B_{dd} = 1$$

$$y_2 = 0.1012238$$

$$B_0^M = \frac{1}{(1.1012238)^2} = 0.8246$$

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Price Condition

- The tree generated from our estimates of u and d is not consistent with the current interest rate structure.
- To make our tree consistent with the term structure, we need to find the S_d value such that when

$$S_u = S_d e^{2\sqrt{hV_e^A}} = S_d e^{2\sqrt{(1)0.0054}}$$

- The value of the two-year bond obtained from the tree is equal to the current equilibrium price of a two-year zero discount bond: $B_0^M = 0.8246$.

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Price Condition

Mathematically, we need to find S_d where:

$$\frac{1}{(1+y_2)^2} = \frac{0.5B_u + 0.5B_d}{1+S_0}$$

$$\frac{1}{(1+y_2)^2} = \frac{0.5[1/(1+S_u)] + 0.5[1/(1/(1+S_d)]}{1+S_0}$$

$$\frac{1}{(1+y_2)^2} = \frac{0.5[1/(1+S_d e^{2\sqrt{hV_e^A}})] + 0.5[1/(1+S_d)]}{1+S_0}$$

In terms of the example, find S_d where:

$$\frac{1}{(1.1012238)^2} = \frac{0.5[1/(1+S_d e^{2\sqrt{0.0054}})] + 0.5[1/(1+S_d)]}{1.10}$$

Solving the above equation for S_d, yields a rate of 9.5%.

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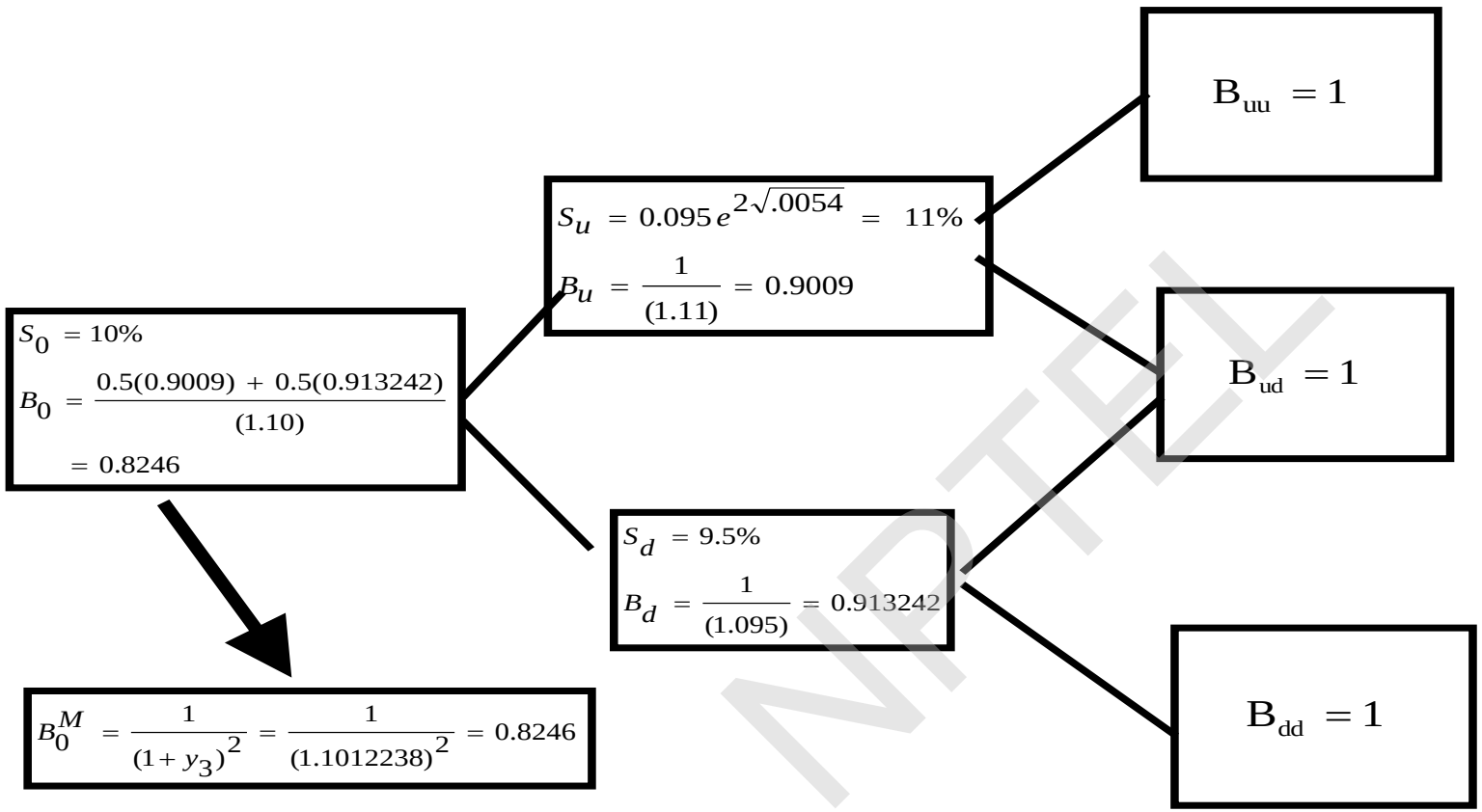
Price Condition

- At S_d = 9.5%, we have a binomial tree of one-year spot rates of S_u = 11% and S_d = 9.5% that simultaneously satisfies our variability condition and price condition.

- That is, the rate is consistent with the estimated volatility of 0.0054 and the current yield curve with one-year and two-year spot rates of 10% and 10.12238%

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Calibration Model



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Price Condition

- The lower rate of 9.5% represents a decline from the current rate of 10%, which is what we tend to expect in a binomial process.
- This is because we have calibrated the binomial tree to a relatively *flat yield curve*.
- If we had calibrated the tree to a *positively sloped yield curve*, then it is possible that both rates next period could be greater than the current rate; although the upper rate will be greater than the lower.
- If we had calibrated the tree to a *negatively sloped curve*, then it is possible that both rates next period could be lower than the current one.

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Calibration Model (Positively sloped Yield Curve)

For example, if the current two-year spot rate were 10.5% instead of 10.12238%, then the equilibrium price of a two-year bond would be 0.8189 and the S_d and S_u values that calibrate the tree to this price and variability of 0.0054 would be 10.20066% and 11.8156%.

$$S_u = 0.102006\, e^{2\sqrt{0.0054}} = 0.118156$$

$$B_u = \frac{1}{1.118156} = 0.8943$$

$$B_{uu} = 1$$

$$B_{ud} = 1$$

$$S_0 = 0.10$$

$$B_0 = \frac{0.5(0.8943) + 0.5(0.9074)}{1.10}$$

$$B_0 = 0.8189$$

$$S_d = 0.1020066$$

$$B_d = \frac{1}{1.1020066} = 0.9074$$

$$B_{dd} = 1$$

$$y_2 = 0.105$$

$$B_0^M = \frac{1}{(1.105)^2} = 0.8189$$

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CONCLUSIONS

- The calibration model generates a binomial tree by finding spot rates that satisfy two conditions: (i) variability condition that governs the relation between the upper and lower spot rates and (ii) ***price condition*** in which the bond value obtained from the tree is consistent with the equilibrium bond price given the current spot yield curve.

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REFERENCES

- Johnson, S. R (2010): Bond Evaluation, Selection and Management, John Wiley & Sons, 2nd Edition.
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Lecture 44: Estimation of Binomial Trees-IV

CONCEPTS COVERED

- Variability and Price condition for multiple periods
- Calibration model for zero coupon, coupon and callable bonds

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KEYWORDS

- Growing the Binomial Tree
- Equilibrium price
- Arbitrage-free price

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Variability Condition: Period 2

- Given our estimated one-year spot rates after one period of 9.5% and 11%, we can now *move to the second period* and determine the tree's three possible spot rates using a similar methodology.
- The variability condition follows the same form as the one period; that is:

$$S_{ud} = S_{dd} e^{2\sqrt{hV_e^A}}$$

$$S_{uu} = S_{ud} e^{2\sqrt{hV_e^A}} = S_{dd} e^{4\sqrt{hV_e^A}}$$

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Price Condition: Period 2

- The price condition requires that the binomial value of a three-year zero coupon bond be equal to the equilibrium price.

- Analogous to the one-period case, this condition is found by solving for the lower rate S_{dd} that, along with the above variability conditions and the rates for S_u and S_d obtained previously, yields a value for a three-year zero-coupon bond that is equal to the price on a three-year zero coupon bond yielding 10.24488%.

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Price Condition: Period 2

Mathematically, find S_{dd} where:

$$\frac{1}{(1+y_2)^3} = \frac{0.5B_u + 0.5B_d}{1+S_0}$$

$$\frac{1}{(1+y_2)^3} = \frac{0.5\left[\frac{0.5B_{uu} + 0.5B_{ud}}{1+S_u}\right] + 0.5\left[\frac{0.5B_{ud} + 0.5B_{dd}}{1+S_d}\right]}{1+S_0}$$

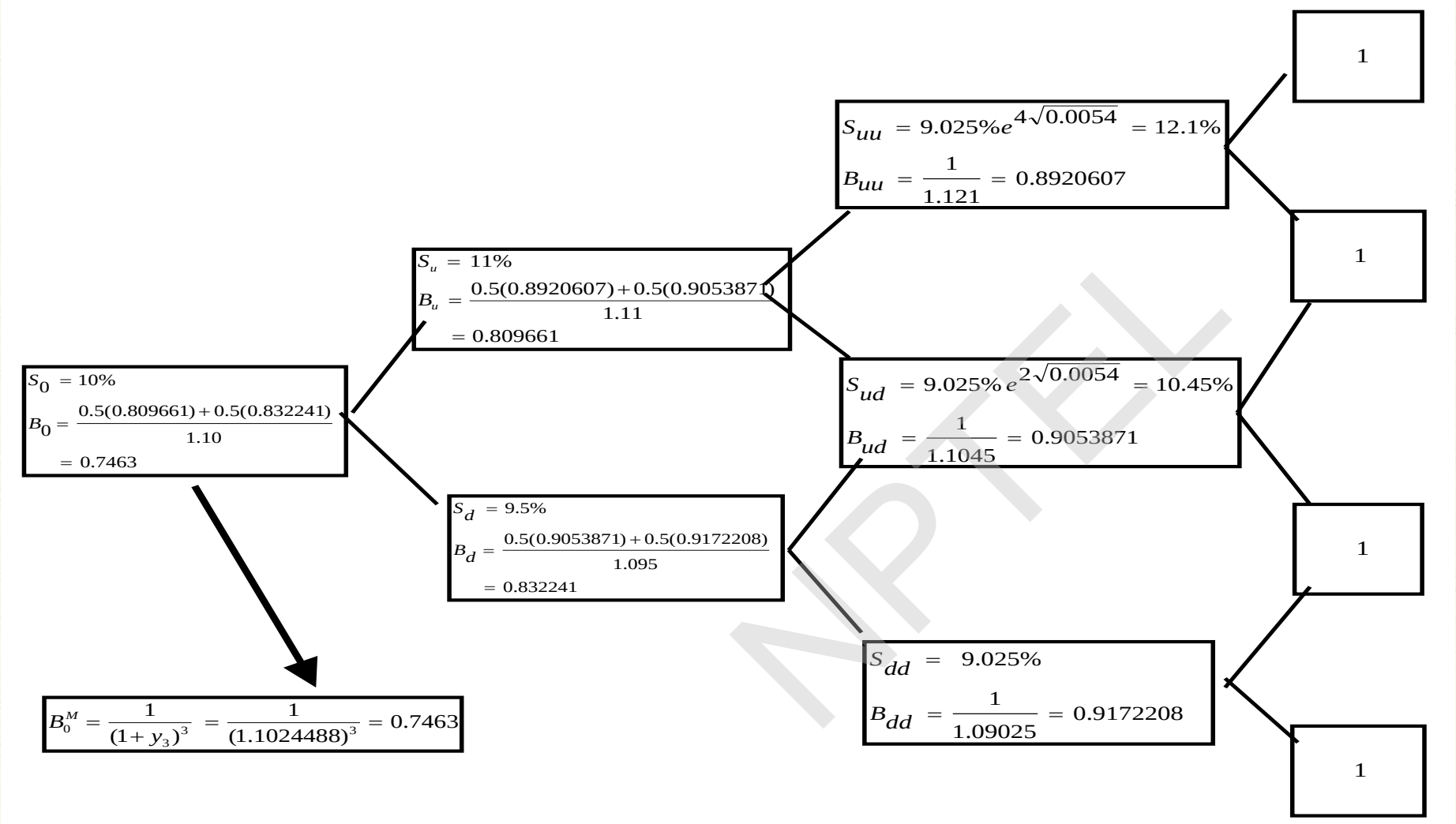
Example, find S_{dd} where

$$\frac{1}{(1.1024488)^3} = \frac{0.5\left[\frac{0.5[1/(1+S_{dd}e^{4\sqrt{0.0054}})] + 0.5[1/(1+S_{dd}e^{2\sqrt{0.0054}})]}{1.11}\right] + 0.5\left[\frac{0.5[1/(1+S_{dd}e^{2\sqrt{0.0054}})] + 0.5[1/(1+S_{dd})]}{1.095}\right]}{1.10}$$

Using an *iterative* (trial and error) approach, we find that a lower rate of S_{dd} = ***9.025%*** yields a binomial value that is equal to the equilibrium price of the three-year bond of 0.7463

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Calibration of Binomial Tree for Three Period Zero Coupon Bond



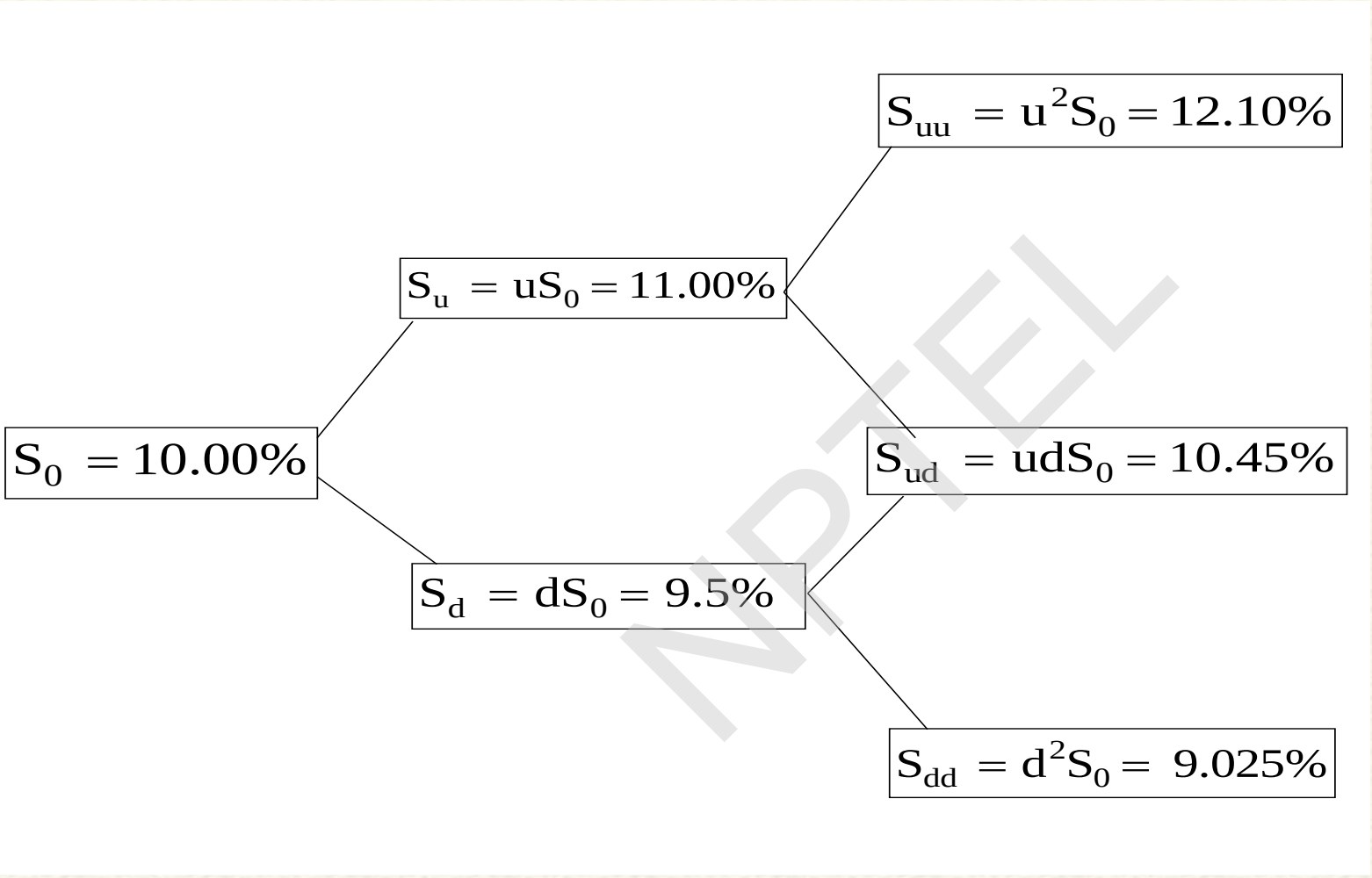
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2-Period Binomial Tree

- The two-period binomial tree is obtained by combining the upper and lower rates found for the first period with the three rates found for the second period.
- This yields a tree that is consistent with the estimated variability condition and with the current term structure of spot rates.

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Calibrated Binomial Tree



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Growing the Binomial Tree

- To grow the tree, we continue with this same process.
- For example, to obtain the four rates in Period 3, we solve for the S_{ddd} that along with the spot rates found previously for periods one and two and the variability relations, yields a value for a four-year PDB that is equal to the equilibrium price.

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Valuation of Coupon Bond

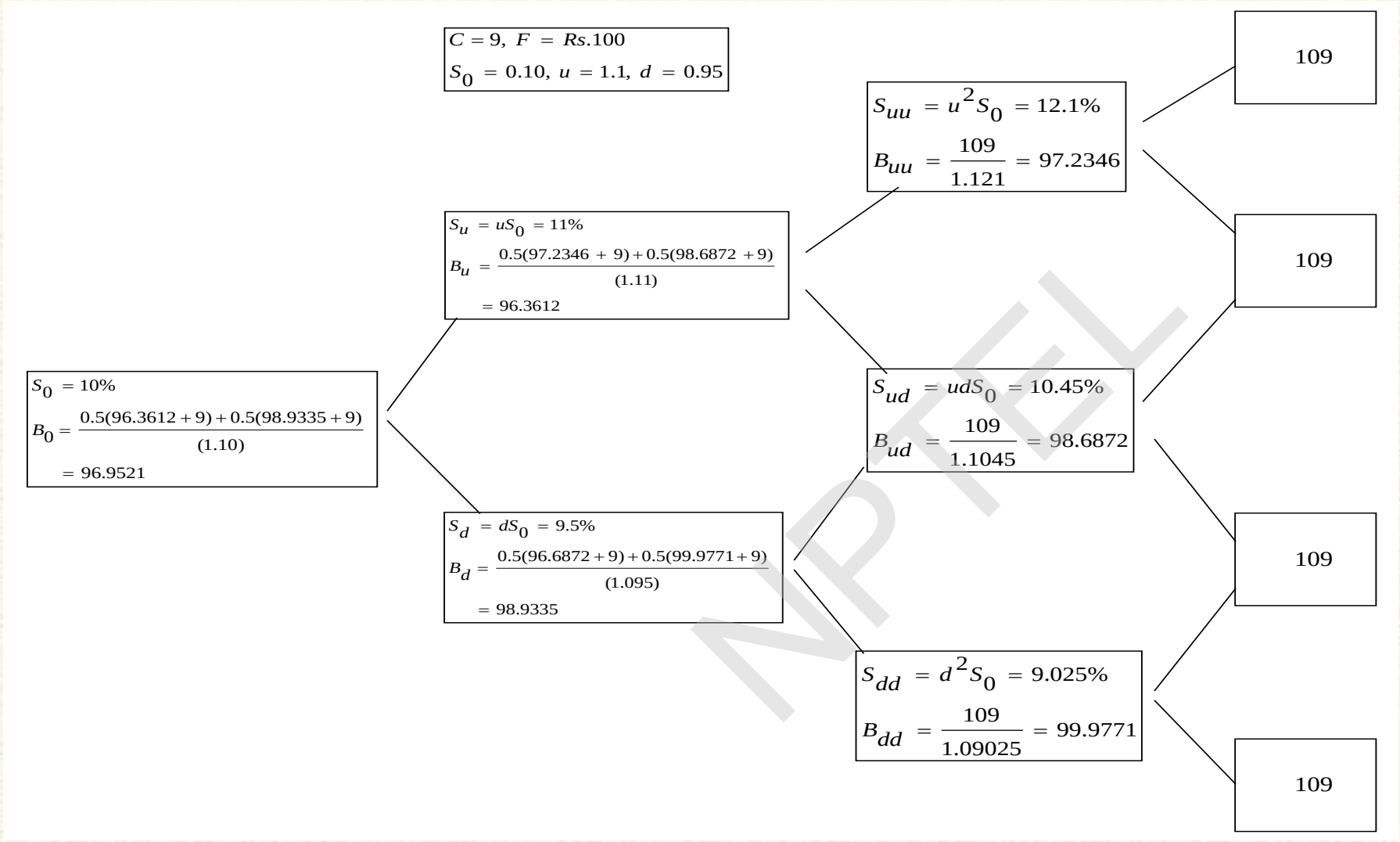
- One of the features of using a calibrated tree to determine bond values is that the tree will yield prices that are equal to the bond's equilibrium price (the price obtained by discounting cash flows by spot rates).
- Example: Discount rate: 1-year: 10%, 2-years: 10.12238%, 3-years: 10.24488%, Coupon: 9%, Face Value: Rs. 100. The value of the bond:

$$B_3^M = \frac{9}{1.10} + \frac{9}{(1.1012238)^2} + \frac{109}{(1.1024488)^3} = 96.9521$$

- The value of a three-year, 9% option-free bond using the tree we derived is also 96.9521.
- Thus, one of the features of the calibrated tree is that it yields values on option-free bonds that are equal to the bond's equilibrium price.

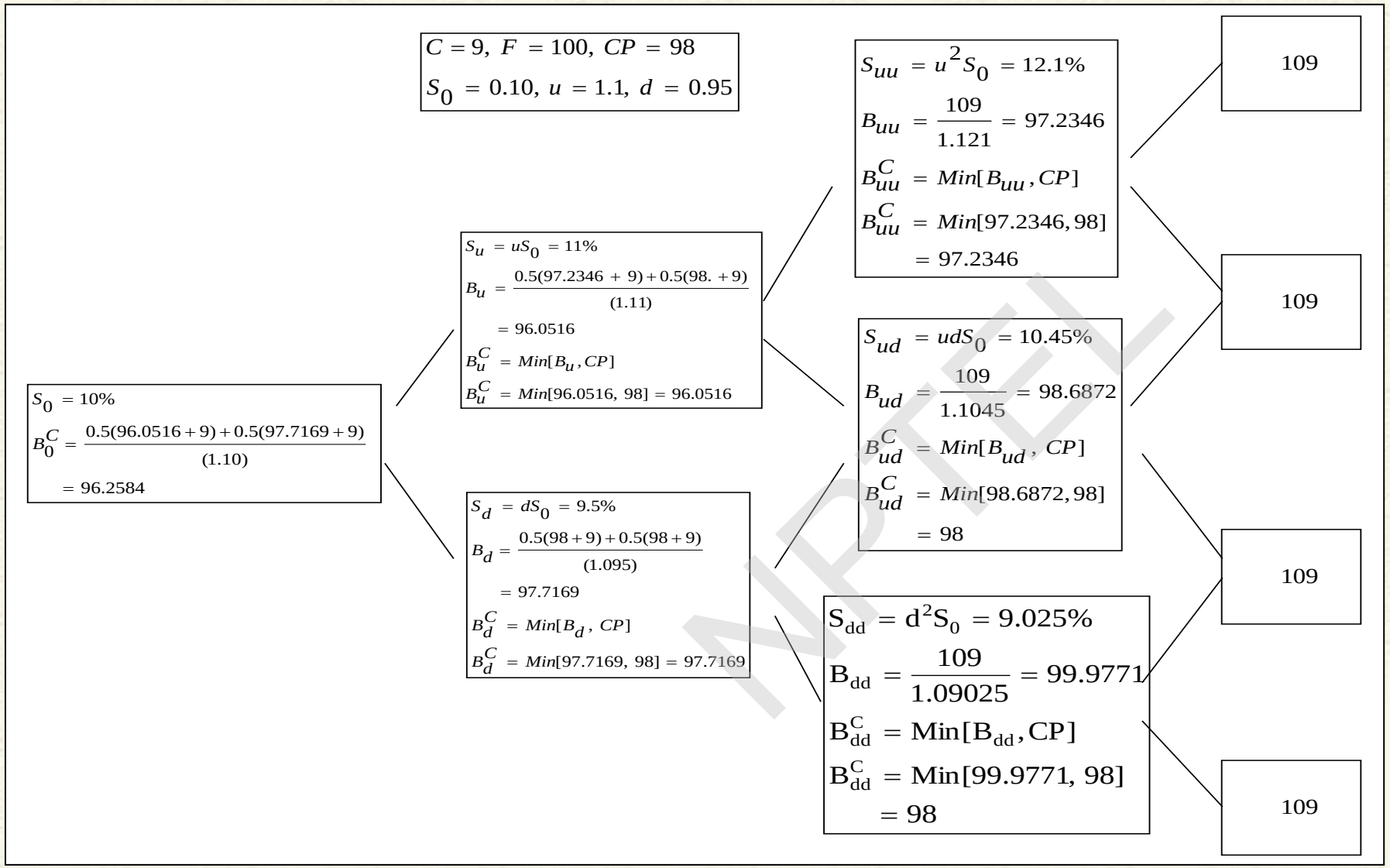
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Calibration Model (Valuation of Coupon Bond)



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Calibration Model for Valuation of Callable Bond



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Option-Free Features

- One of the features of the calibration model is that it prices a bond equal to its equilibrium price.
- A bond's equilibrium price is an arbitrage-free price.
- If the market does not price the bond at its equilibrium value, then arbitrageurs would be able to realize a riskless return either by buying the bond, stripping it into a number of zero discount bonds, and selling them, or by buying a portfolio of zero discount bonds, bundling them into a coupon bond, and selling it.

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Option-Free Features

- In general, a security can be valued by arbitrage by pricing it to equal the value of its replicating portfolio: a portfolio constructed so that it has the same cash flows.
- The replicating portfolio of a coupon bond is the portfolio of zero discount bonds.
- One of the important features of the calibration model is that it yields prices on option-free bonds that are arbitrage free.
- In addition to satisfying an arbitrage-free condition on option-free bonds, the calibration model also values a bond's embedded options as arbitrage-free prices.

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CONCLUSIONS

- One of the features of using a calibrated tree to determine bond values is that the tree will yield prices that are equal to the bond's equilibrium price
- A bond's equilibrium price is an arbitrage-free price. If the market does not price the bond at its equilibrium value, then arbitrageurs would be able to realize a riskless return

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- Johnson, S. R (2010): Bond Evaluation, Selection and Management, John Wiley & Sons, 2nd Edition.
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NPTEL ONLINE CERTIFICATION COURSES

Management of Fixed Income Securities

Prof. Jitendra Mahakud
Department of Humanities and Social Sciences
Indian Institute of Technology Kharagpur

Lecture 45: Estimation of Binomial Trees-V

CONCEPTS COVERED

- Option adjusted spread analysis
- Effective duration
- Effective convexity

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KEYWORDS

- **Option adjusted spread**
- **Path analysis**

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Option-Adjusted Spread

- The binomial tree also can be used to estimate the ***option-adjusted spread***
- The simplest way to estimate the option spread is to estimate the YTM for a bond with an option given the bond's values as determined by the binomial model, then subtract that rate from the YTM of an otherwise identical option-free bond.
- ***Example:*** In the previous example the value of the three-year, 9% callable was 96.2584, whereas the equilibrium price of the noncallable was 96.9521

$$Option-Free\ Bond:\ 96.9521=\frac{9}{1+YTM}+\frac{9}{(1+YTM)^2}+\frac{109}{(1+YTM)^3}$$

$$\Rightarrow YTM^{NC}=10.2306\%$$

$$Callable\ Bond:96.2584=\frac{9}{1+YTM}+\frac{9}{(1+YTM)^2}+\frac{109}{(1+YTM)^3}$$

$$\Rightarrow YTM^{C}=10.51832\%$$

$$Option\ Spread=YTM^{C}-YTM^{NC}=10.51832\%-10.2306\%=0.28772\%$$

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Option-Adjusted Spread

- The problem with using this approach to estimate the spread is that not all of the possible cash flows of the callable bond are considered.
- In three of the four interest rate scenarios, for example, the bond could be called, changing the cash flow pattern from three periods of 9, 9, and 109 to two periods of 9 and 107.
- An alternative approach that addresses this problem is the **option-adjusted spread (OAS) analysis**.
- The objective of option-adjusted spread analysis is to solve for the option spread, k, that makes the average of the present values of the bond's cash flows from all of the possible interest rate paths equal to the bond's market price.

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Option-Adjusted Spread Analysis

- The ***first step*** is to specify the cash flows and spot rates for each path.
- In the case of the three-year bond valued with a two-period binomial interest rate tree, there are four possible paths.

Path/ Time	1 S	1 CF	2 S	2 CF	3 S	3 CF	4 S	4 CF
0	0.10	-	0.10	-	0.10	-	0.10	-
1	0.0950	9	0.095	9	0.11	9	0.11	9
2	0.09025	107	0.1045	107	0.1045	107	0.121	9
3	-		-		-		-	109

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Option-Adjusted Spread Analysis

- The ***next step*** is to determine the appropriate two-year spot rates (y_2) and three-year rates (y_3) to discount the cash flows.
- These rates can be found using the geometric mean and the one-year spot rates from the tree.

Path 1

$y_1 = 0.10$

$y_2 = [(1.10)(1.095)]^{1/2} - 1 = 0.097497$

$y_3 = [(1.10)(1.095)(1.09025)]^{1/3} - 1 = 0.095076$

Path 3

$y_1 = 0.10$

$y_2 = [(1.10)(1.11)]^{1/2} - 1 = 0.104989$

$y_3 = [(1.10)(1.11)(1.1045]^{1/3} - 1 = 0.104826$

Path 2

$y_1 = 0.10$

$y_2 = [(1.10)(1.095)]^{1/2} - 1 = 0.097497$

$y_3 = [(1.10)(1.095)(1.1045]^{1/3} - 1 = 0.099826$

Path 4

$y_1 = 0.10$

$y_2 = [(1.10)(1.11)]^{1/2} - 1 = 0.104989$

$y_3 = [(1.10)(1.11)(1.121]^{1/3} - 1 = 0.110300$

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Option-Adjusted Spread Analysis

- The ***last step*** is to is to solve for the k that makes the average present values of the paths equal to the callable bond's market price, B^M_0.

$$B_0^M = (1/4)\left\{ \begin{array}{l} \left[\frac{9}{(1+0.10+k)} + \frac{107}{(1+0.097497+k)^2}\right] \\ + \left[\frac{9}{(1+0.10+k)} + \frac{107}{(1+0.097497+k)^2}\right] \\ + \left[\frac{9}{(1+0.10+k)} + \frac{107}{(1+0.104989+k)^2}\right] \\ + \left[\frac{9}{(1+0.10+k)} + \frac{9}{(1+0.104989+k)^2} + \frac{109}{(1+0.110300+k)^3}\right] \end{array} \right\}$$

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Option-Adjusted Spread Analysis

- If the market price is equal to the binomial value we obtained using the calibration model, then the option spread, k, is equal to zero.
- This reflects the fact that we have calibrated the tree to the yield curve and have considered all of the possibilities.
- In practice, though, we do not expect the market price to equal the binomial value.
- If the market price is below the binomial value, then k will be positive.

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Effective Duration and Convexity

- The option features of a bond can have a significant impact on the bond's duration, as well as its convexity

- For example: When a bond has a call option, a rate decrease can lead to an early call, which shortens the life of the bond and lowers its duration, while an interest rate increase tends to lengthen the expected life of the bond causing its duration to increase.

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Effective Duration and Convexity

- The duration and convexity of bonds with embedded option features can be estimated using a binomial tree
- The effective duration and convexity measures are:

$$\text{Effective Duration} = \frac{B_- - B_+}{2(B_0)(\Delta y)}$$

$$\text{Effective Convexity} = \frac{B_+ + B_- - 2B_0}{(B_0)(\Delta y)^2}$$

where:

B_- = price associated with a small decrease in rates

B_+ = price associated with small increase in rates

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Effective Duration and Convexity

- The binomial tree calibrated to the yield curve can be used to estimate B_0, B_- and B_+:
  - B_0 can be defined by the current yield curve and calibrated tree.
  - B_- can be estimated by allowing for a small equal decrease in each of the yield curve rates and then using the tree calibrated to the new rates to find the price.
  - B_+ can be estimated in a similar way by allowing for a small equal increase in the yield curve's rates and then estimating the bond price using the tree calibrated to these higher rates.

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CONCLUSIONS

- The objective of option-adjusted spread analysis is to solve for the option spread that makes the average of the present values of the bond's cash flows from all of the possible interest rate paths equal to the bond's market price.
- The option features of a bond can have a significant impact on the bond's duration, as well as its convexity

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REFERENCES

- Johnson, S. R (2010): Bond Evaluation, Selection and Management, John Wiley & Sons, 2nd Edition.
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